

Cyclic Cohomology Groups of Some Self-similar Sets

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Abstract

We define a variant of the Young integration on some kinds of self-similar sets which are called *cellular self-similar sets*. This variant is an analogue of the Young integration defined on the unit interval. We give the criteria of the variant on cellular self-similar sets, and also show that the variant is a cyclic 1-cocycle of the algebra of complex-valued α -Hölder continuous functions on the cellular self-similar sets. This suggests that the cocycle is a variant of currents.

1 Introduction

Fractal sets introduced by Mandelbrot [16] are complex-behaved spaces difficult to analyse. For instance, for the Cantor sets, their Hausdorff dimensions are different, although all of them are topologically isomorphic. Since the Hausdorff dimension is an invariant of fractal sets which is stable under bi-Lipschitz transformations, it is hard to say that the (co)homology theories which are homotopy invariant can capture deeper topological quantities of fractal sets.

Connes introduced cyclic cohomology theory [4], which turns out to be a generalisation of the de Rham homology theory. He proposed Quantised calculus in [5] and exploits the Dixmier trace as a non-smooth analogue of the integration on manifolds. Namely, he applied it to the Cantor sets and succeeded to recover their Minkowski contents as the value of a certain Dixmier trace. This result suggests that cyclic cohomology theory could be one of suitable methods to study fractal sets.

Another approach to analyse some fractal sets was proposed by Moriyoshi and Natsume [18]. For the Sierpinski gasket SG , they exploit the algebra $C^1(SG)$ of Lipschitz functions on SG to construct a cyclic 1-cocycle ϕ of $C^1(SG)$. They also show that, when Lipschitz functions are seen as 1-Hölder continuous functions, the regularity $\alpha = 1$ of $C^1(SG)$ for the well-definedness of ϕ can be reduced to the half of the Hausdorff dimension $\dim_H(SG)/2$.

In order to define the cyclic 1-cocycle ϕ , Moriyoshi and Natsume exploit the Young integration on the unit interval I . It has following properties, which explain a reason why Hölder continuous functions are used and the Hausdorff dimension arises in the results of [18]: the Young integration on the unit interval was developed in [26]. This is a bilinear function from the product $W_\alpha \times W_\beta$ of the Wiener classes such that $\alpha + \beta > 1 = \dim_H(I)$ to complex numbers:

$$Y : W_\alpha \times W_\beta \rightarrow \mathbb{C}.$$

Especially, if we restrict the domain of Y to the algebra $C^\alpha(I)$ of α -Hölder continuous functions on I , the map Y is well-defined for $2\alpha > 1 = \dim_H(I)$. We note that $C^\alpha(I)$ contains the algebra $C^\infty(I)$ of smooth functions on I , and then, for $f, g \in C^\infty(I)$ we further get

$$Y(f, g) = \int_I f dg.$$

In this sense, the Young integration may be considered as a generalisation of the integration of differential 1-forms, and does make it possible to integrate some non-smooth functions. Moreover, for a Jordan curve C composed of a finite number of unit intervals, the Young integration Y along C turns out to be a cyclic 1-cocycle if $2\alpha > 1 = \dim_H(C)$:

$$Y : C^\alpha(C) \times C^\alpha(C) \rightarrow \mathbb{C}.$$

All those results above motivate us to extend the cyclic 1-cocycle ϕ of $C^1(SG)$ onto a certain class of fractal sets by exploiting the Young integration.

In this paper, we extend the cocycle of the Sierpinski gasket defined in [18] to a certain class of self-similar sets by exploiting the Young integration, and show that the cocycles can be applied to some examples. More detailed and precise statements are given as follows.

We first define cellular self-similar sets, the preliminary notions of which are given in Section 2.2 below. Cellular self-similar sets $K_{|X|}$ are self-similar sets that are based on linear cell complexes, and the unit interval is a prototype of cellular self-similar sets. The precise definition is as follows:

Definition 1.1 (Definition 3.1). Let $|X|$ be a 2-dimensional finite convex linear cell complex and $\{F_j\}_{j \in S}$ a set of similitudes $F_j : |X| \rightarrow |X|$ indexed by a finite set S . We also let $|X_1| = \bigcup_{j \in S} F_j(|X|)$. The triple $(|X|, S, \{F_j\}_{j \in S})$ is called a *cellular self-similar structure* if it satisfies

- a) $\partial|X| \subset \partial|X_1|$, and
- b) $\text{int } F_i(|X|) \cap \text{int } F_j(|X|) = \emptyset$, for all $i \neq j \in S$.

Let $(|X|, S, \{F_j\}_{j \in S})$ be a cellular self-similar structure. Then $(|X|, S, \{F_j\}_{j \in S})$ yields a sequence $\{|X_n|\}_{n \in \mathbb{N}}$ of 2-dimensional cell complexes, and, by Theorem 2.5 below, the sequence gives rise to the cellular self-similar set $K_{|X|}$ with respect to $(|X|, S, \{F_j\}_{j \in S})$. For every $n \in \mathbb{N}$, $|X_n|$ is subdivided into a simplicial complex $|X_n^s|$ by Lemma 1 of Chapter 1 in [27]. From this simplicial complex, we get 1-chains b_n, I_n, o_n and $I_n \setminus I_{n-1} \in \tilde{S}_1(X_n^s; \mathbb{C})$ whose geometric incarnations are subspaces of 1-skelton of $|X_n^s|$; see Section 3.1 for the details.

On the other hand, the algebra $C^\alpha(|X_n^s|)$ of complex-valued α -Hölder continuous functions defined on $|X_n^s|$ is a subspace of the function space $F^0(|X_n^s|; \mathbb{C}) = \{f : |X_n^s| \rightarrow \mathbb{C}\}$ as a \mathbb{C} -vector space. Therefore, $C^\alpha(|X_n^s|)$ can generate a \mathbb{C} -vector space $C^{\alpha,1}(|X_n^s|)$ with the differential and cup product of the Alexander-Spanier cochain complex [21], which is a subspace of $\text{Hom}_{\mathbb{C}}(\tilde{S}_1(|X_n^s|), \mathbb{C})$; see Section 3.2 for the details. For f and $g \in C^\alpha(K_{|X|})$ we have a cochain $f \smile \delta(g) - g \smile \delta(f) \in C^{\alpha,1}(|X_n^s|)$ for any $n \in \mathbb{N}$, which is denoted by $\omega_n(f, g)$. Finally we set $\phi_n(f, g)$ as $\omega_n(f, g)(I_n)$ and call the sequence $\{\phi_n(f, g)\}_{n \in \mathbb{N}}$ the cyclic quasi-1-cocycle of f and g , the definition of which is given in Section 3.2.

The first main theorem states that if $2\alpha > \dim_H(K_{|X|})$, we can define a bilinear map $\phi : C^\alpha(K_{|X|}) \times C^\alpha(K_{|X|}) \rightarrow \mathbb{C}$ by taking the limit of the cyclic quasi-1-cocycle $\{\phi_n(f, g)\}_{n \in \mathbb{N}}$.

This implies that the bilinear map ϕ may be seen as a generalisation of the classical Young integration of the unit interval.

Theorem 1.2 (Theorem 3.8, Existence theorem). *Let $(|X|, S, \{F_j\}_{j \in S})$ be a cellular self-similar structure with $\#S \geq 2$ and $K_{|X|}$ the cellular self-similar set with respect to $(|X|, S, \{F_j\}_{j \in S})$. We also let $C^\alpha(K_{|X|})$ be the algebra of α -Hölder continuous functions on $K_{|X|}$. If $2\alpha > \dim_H(K_{|X|})$, then the cyclic quasi-1-cocycle $\{\phi_n(f, g)\}$ is a Cauchy sequence for any $f, g \in C^\alpha(K_{|X|})$.*

The map ϕ is originally defined by Moriyoshi and Natsume [18] for the algebra $C^{Lip}(SG)$ of complex-valued Lipschitz functions on the Sierpinski gasket SG , and the construction is based on the classical Young integration on the unit interval. The authors use the simplexes I_n to prove the existence of the cyclic cocycle of the Sierpinski gasket. An obstacle to extend the construction to cellular self-similar sets is that, for each $n \in \mathbb{N}$, the lengths of 1-simplices belonging in $|X_n|$ are not equal. The key technical ingredient to overcome the difficulty is the existence of 2-dimensional simplicial complex $|K_{n,n+1}^s|$ whose boundary is a disjoint union of $\partial(|X_n|)$ and $\partial(|X_{n+1}|)$. By properties of cellular self-similar sets, we can prove that lengths of 1-simplices of $|K_{n,n+1}|$ have an upper bound which tends to 0 as $n \rightarrow \infty$. This property plays a crucial role to prove that $\{\phi_n(f, g)\}_{n \in \mathbb{N}}$ is a Cauchy sequence.

The proof of the above theorem immediately yields the following corollary. This proves that the bilinear map ϕ is a non-commutative representation of the Young integration.

Corollary 1.3 (Corollary 3.10). *For any $f, g \in C^\alpha(K_{|X|})$ with $2\alpha > \dim(K_{|X|})$, we have*

$$\phi(f, g) = 2 \int_{\partial|X|}^{Young} f dg = 2 \cdot (\text{Young integration of } f \text{ and } g \text{ along } \partial|X|).$$

In particular, if $|X| \neq |X_1|$, for 1 and $x := id \in C^\alpha(K_{|X|})$, we get

$$\phi(1, x) = 2 \int_{\partial|X|}^{Young} dx = 2 \cdot (\text{length of } \partial|X|).$$

After we define ϕ of the cellular self-similar set $K_{|X|}$, we prove that ϕ is a cyclic 1-cocycle of $C^\alpha(K_{|X|})$ and represents a nontrivial element in the first cyclic cohomology group $HC^1(C^\alpha(K_{|X|}))$. This theorem shows that ϕ may be seen as a non-commutative generalisation of the integration on manifolds.

Theorem 1.4 (Theorem 3.11). *Under the assumption of the existence theorem:*

- a) The bilinear map ϕ is a cyclic 1-cocycle of $C^\alpha(K_{|X|})$.*
- b) If $|X| \neq |X_1|$, the cocycle ϕ represents a non-trivial element $[\phi]$ in $HC^1(C^\alpha(K_{|X|}))$.*

For the proof of the first statement, we need to use the Leibniz rule of the cup product defined on the Alexander-Spanier cochain complex. Corollary 1.3 immediately completes the proof of the second statement since $1 \otimes x$ represents an element in the Hochschild homology group.

By Theorem 1.4, we find that the cocycle ϕ has the following additional properties: ϕ can detect the Hausdorff dimensions of cellular self-similar sets and distinguish them by their dimensions. For instance, we get the cocycles ϕ of the Sierpinski gasket SG and the Sierpinski

carpet SC , whose thresholds of the well-definedness are different. Namely, their thresholds are $\dim_H(SG) = \log_2 3$ and $\dim_H(SC) = \log_3 8$. Since bi-Lipschitz transformations preserve the Hausdorff dimension, the cocycles can prove that SG and SC are not bi-Lipschitz homeomorphic. Moreover, if we have a bi-Lipschitz transformation between cellular self-similar sets $K_{|X|}$ and $K_{|X'|}$, the algebra $C^\alpha(K_{|X|})$ is isomorphic to $C^\alpha(K_{|X'|})$. Therefore, we further get the following property: the cocycle ϕ is invariant under bi-Lipschitz transformations. After the proof of the main results, we apply the results to some examples, and extend the cocycle to some variants of cellular self-similar sets.

Conventions

We assume that algebras have unit unless otherwise stated, and all base rings of algebras is the field \mathbb{C} of complex numbers. The Euclidean space \mathbb{R}^n is endowed with the standard Euclidean metric.

- $\otimes = \otimes_{\mathbb{C}}$.
- $\mathbb{Z}_{\geq 0} = \mathbb{N} \cup \{0\}$.
- ∂X : the boundary of a topological space X .
- $C^\alpha(X)$: the algebra of complex-valued α -Hölder continuous functions on a metric space X whose sum and multiplication are given by the pointwise sum and multiplication.
- $C^{Lip}(X)$: the algebra of complex-valued Lipschitz functions on a metric space X whose sum and multiplication are given by the pointwise sum and multiplication.

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2 Preliminaries

In this section, we recall the key materials for the main theorems, the Young integral, self-similar sets.

2.1 Young Integral

We begin with a quick review of the Young integral basically following [26] except the slight changes of the notation.

Let I be the unit interval $[0, 1]$ and f, g complex-valued functions defined on I . We make a subdivision χ of I

$$0 = x_0 < x_1 < \cdots < x_{n-1} < x_n = 1$$

and define

$$F(\chi) = \sum_{i=1}^n f(x_i)(g(x_i) - g(x_{i-1})).$$

Then $F(\chi)$ can be also written as

$$F(\chi) = \sum_{0 < i \leq j \leq n} \delta(f)(x_{i-1}, x_i) \cdot \delta(g)(x_{j-1}, x_j) + f(0)(g(1) - g(0)).$$

Here $\delta(f)(x_{i-1}, x_i)$ denotes $f(x_i) - f(x_{i-1})$, and this notation is similar to the differential of the Alexander-Spanier cohomology theory; see also Chapter 6.4 in [21]. We also let $\alpha, \beta > 0$, and denote by $S_{\alpha, \beta}[0, 1] = S_{\alpha, \beta}[0, 1; f, g]$ the upper bound of

$$\left(\sum_i |\delta(f)(x_{i-1}, x_i)|^{\frac{1}{\alpha}} \right)^{\alpha} \left(\sum_i |\delta(g)(x_{i-1}, x_i)|^{\frac{1}{\beta}} \right)^{\beta}$$

for every subdivision of I . Following lemmas of [26], if $\alpha + \beta > 1$ and $\xi \in [0, 1]$ is a division point of χ , we have

$$\left| F(\chi) - f(\xi)(g(1) - g(0)) \right| \leq (1 + \zeta(\alpha + \beta)) \cdot S_{\alpha, \beta}[0, 1],$$

where $\zeta(\alpha + \beta)$ denotes the zeta function of $\alpha + \beta$.

This inequality yields to a more general inequality for the sum associated to χ : for the given subdivision χ , let a point $x_{i-1} \leq \xi_i \leq x_i$ for each i , and applying this inequality for each interval $[x_{i-1}, x_i]$ and summing up, we get

$$\left| F(\chi) - \sum_{i=1}^n f(\xi_i)(g(x_i) - g(x_{i-1})) \right| \leq \{1 + \zeta(\alpha + \beta)\} \cdot \sum_{i=1}^n S_{\alpha, \beta}[x_{i-1}, x_i; f, g].$$

Moreover if we have another subdivision χ' of I and subdivision points $x_{j-1} \leq \xi'_j \leq x_j$, then

$$\begin{aligned} & \left| \sum_{i=1}^n f(\xi_i)(g(x_i) - g(x_{i-1})) - \sum_{j=1}^m f(\xi'_j)(g(x'_j) - g(x'_{j-1})) \right| \\ & \leq \{1 + \zeta(\alpha + \beta)\} \cdot \left\{ \sum_{i=1}^n S_{\alpha, \beta}[x_{i-1}, x_i; f, g] + \sum_{j=1}^m S_{\alpha, \beta}[x'_{j-1}, x'_j; f, g] \right\}. \end{aligned}$$

Definition 2.1. We say that the *Stieltjes integral*

$$\int_0^1 f dg$$

exists in the Riemann sense with the value J , if there exist $J \in \mathbb{C}$ and a function $\varepsilon_\delta > 0$ with respect to the variable $\delta > 0$ such that $\varepsilon_\delta \rightarrow 0$ as $\delta \rightarrow 0$, and if all the segments $[x_{i-1}, x_i]$ of a subdivision χ have lengths less than $\delta > 0$, then

$$\left| J - \sum_i f(\xi_i)(g(x_i) - g(x_{i-1})) \right| < \varepsilon_\delta.$$

We observe that, for the integrability in the Riemann sense, it is sufficient that the difference of any of two sums of the formula $\sum_i f(\xi_i)(g(x_i) - g(x_{i-1}))$ of Definition 2.1, for each of which the length of $[x_{i-1}, x_i]$ is less than δ , is less than ε_δ . By the inequality just before Definition 2.1, this is the case if for some $\alpha, \beta > 0$ such that $\alpha + \beta > 1$ we have

$$\sum_{i=1}^n S_{\alpha, \beta}[x_{i-1}, x_i; f, g] < \varepsilon_\delta.$$

For the existence of the integrability, we define $W_\alpha(\delta)$ to be the set of functions such that the value $V_\alpha^{(\delta)}(f)$ defined below has an upper bound:

$$V_\alpha^{(\delta)}(f) = \sup_{|\chi| \leq \delta} \left\{ \left(\sum_i |f(x_i) - f(x_{i-1})|^{\frac{1}{\alpha}} \right)^\alpha \right\} < \infty.$$

Here $|\chi|$ denotes the maximum length of the intervals of χ , and the supremum runs over all subdivisions χ such that $|\chi|$ is less than or equal to δ . Finally we define the *Wiener class* W_α to be the set of functions f such that $V_\alpha^{(\delta)}(f)$ with respect to the variable δ has an upper bound.

Theorem 2.2 (Theorem on Stieltjes integrability). *If $f \in W_\alpha$ and $g \in W_\beta$ where $\alpha, \beta > 0$ and $\alpha + \beta > 1$, have no common discontinuities, their Stieltjes integral exists in the Riemann sense.*

The Wiener class W_α is closed under the pointwise sum and scalar multiplication for $0 < \alpha < 1$. Therefore, if we regard the integration as a function from $W_\alpha \times W_\alpha$ to \mathbb{C} , this function turns out to be a bilinear function. On the other hand, it is clear from the definition that the set $C^\alpha(I)$ of complex-valued α -Hölder continuous functions defined on I is a subset of W_α . Moreover, $C^\alpha(I)$ is closed under the pointwise multiplication in addition to the pointwise sum and scalar multiplication. The integration restricted to $C^\alpha(I)$ is also referred to as the *Young integration*.

Remark 2.3. The Young integration is a special case of the Riemann-Stieltjes integration.

2.2 Self-similar Sets and Hausdorff Dimension

In this subsection we briefly recall the definition of self-similar sets and the Hausdorff dimension. This subsection basically follows [12]. At the end of this subsection, we give some examples of self-similar sets. We first begin with the definition of some maps from a metric space (X, d) to itself.

Definition 2.4. Let (X, d) be a metric space.

- a) A map $F : X \rightarrow X$ is a *contraction* if there exists $0 < r \leq 1$ such that $d(F(x), F(y)) \leq r \cdot d(x, y)$ for any $x, y \in X$. The real number r is called the *contraction ratio*.
- b) A contraction $F : X \rightarrow X$ is a *similitude* if $d(F(x), F(y)) = r \cdot d(x, y)$ for any $x, y \in X$. We call r the *similarity ratio*.

For a finite set $\{F_j\}_{j \in S}$ of contractions defined on a complete metric space, there exists a unique compact subspace that is characterised by $\{F_j\}_{j \in S}$. Here is the precise statement of the existence of self-similar sets:

Theorem 2.5. *Let X be a complete metric space. We also let S be a finite set and $F_i : X \rightarrow X$ contractions indexed by S . We call the triple $(X, S, \{F_j\}_{j \in S})$ an iterated function system or IFS. Then, there exists a unique non-empty compact subset K_X of X that satisfies*

$$K_X = \bigcup_{j \in S} F_j(K_X).$$

The compact set K_X is called the *self-similar set with respect to $(X, S, \{F_j\}_{j \in S})$* .

Remark 2.6. In some literature the terminology “self-similar set” is used in a restricted sense. For instance, Hutchinson introduces the notion of “self-similar set” for a finite set of similitudes [11], and self-similar sets defined in Theorem 2.5 are referred to as *attractors* or *invariant sets*; see Section 9.1 in [7]. We employ Hutchinson’s definition of self-similar sets in the last section to define cellular self-similar sets, the definition of which is given in Section 3.1 below.

For later use, we include an outline of a proof of the above theorem. The proof is based on the following theorem.

Theorem 2.7 (Contraction principle). *Let (X, d) be a complete metric space and $F : X \rightarrow X$ a contraction with respect to the metric. Then there exists a unique fixed point of F , in other words, there exists a unique solution to the equation $F(x) = x$. Moreover if x_* is the fixed point of F , then $\{F^n(a)\}_{n \geq 0}$ converges to x_* for all $a \in X$ where F^n is the n -th iteration of F .*

Let (X, d) be a metric space and $K(X)$ the set of non-empty compact subsets of X . We define the *Hausdorff metric* δ on $K(X)$ by

$$\delta(A, B) = \inf\{r > 0 : U_r(A) \subset B \text{ and } U_r(B) \subset A\},$$

where $U_r(A) = \{x \in X : d(x, A) \leq r\}$.

Lemma 2.8. *The pair $(K(X), \delta)$ forms a metric space. Moreover, if X is complete, $(K(X), \delta)$ is also complete.*

We now assume that the metric space (X, d) is complete. Define $F(A) = \bigcup_{j \in S} F_j(A)$ for $A \subset X$, and then $F : K(X) \rightarrow K(X)$ is a contraction with respect to the metric δ . Therefore, by applying Theorem 2.7 to $(K(X), \delta)$ and F , we get the self-similar set K_X with respect to $(X, S, \{F_j\}_{j \in S})$.

We next define the Hausdorff dimension, which plays a key role to define cyclic cocycles on cellular self-similar sets, the definition of which are given in Section 3 below.

Definition 2.9. Let (X, d) be a metric space. We also let $s > 0$ and $\delta > 0$. For any bounded set $A \subset X$, we define

$$\mathcal{H}_\delta^s(A) = \inf \left\{ \sum_{i \geq 1} \text{diam}(E_i)^s : A \subset \bigcup_{i \geq 1} E_i, \text{diam}(E_i) \leq \delta \right\}.$$

Here the infimum runs over all the coverings $\{E_i\}$ of A , which consist of sets, and $\text{diam}(E_i)$ denotes the diameter of E_i . Also we define

$$\mathcal{H}^s(A) = \limsup_{\delta \downarrow 0} \mathcal{H}_\delta^s(A),$$

and we call \mathcal{H}^s the s -dimensional Hausdorff measure of (X, d) .

Remark 2.10. The s -dimensional Hausdorff measure is a complete Borel measure.

The measure detects a critical point of the given subset.

Lemma 2.11. For any subset $E \subset X$, we have

$$\sup \{s \in \mathbb{R} \mid \mathcal{H}^s(E) = \infty\} = \inf \{s \in \mathbb{R} \mid \mathcal{H}^s(E) = 0\}.$$

Definition 2.12. The real number which satisfies Lemma 2.11 is called the *Hausdorff dimension* of E , and it is denoted by $\dim_H(E)$.

In general, it is difficult to calculate the Hausdorff dimension. Namely, the Hausdorff dimensions of a few self-similar sets have been computed. However, if we have a self-similar set K_X with respect to an IFS $(X, S, \{F_j\}_{j \in S})$ such that contractions are similitudes and the similitudes have “small” enough intersections, then there exists a useful way to compute the Hausdorff dimension of K_X .

Theorem 2.13. [17, Theorem II] Let X be a compact subspace in \mathbb{R}^n and $\{F_j : \mathbb{R}^n \rightarrow \mathbb{R}^n\}_{j \in S}$ a finite set of similitudes indexed with a finite set S . Suppose that the self-similar set K_X with respect to the IFS $(X, S, \{F_j\}_{j \in S})$ satisfies the open set condition, i.e., there exists a bounded non-empty open set $O \subset \mathbb{R}^n$ such that

$$\bigcup_{j \in S} F_j(O) \subset O \quad \text{and} \quad F_i(O) \cap F_j(O) = \emptyset \quad \text{for any } i \neq j \in S.$$

Then the Hausdorff dimension $\dim_H(K_X)$ of the self-similar set K_X is the unique real number α such that the following relation holds

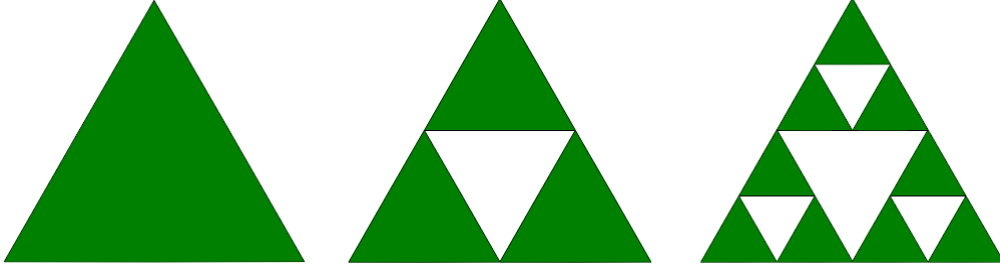
$$\sum_{j \in S} r_j^\alpha = 1.$$

Here r_j denotes the similarity ratio of F_j .

Example 2.14. In this example we give examples of self-similar sets and their dimensions. For later use, we explain contractions of each self-similar set and give an IFS $(X, S, \{F_j\}_{j \in S})$ that gives rise to the self-similar set. We also provide figures for each self-similar set, that correspond to X , $F(X) (= \bigcup_{j \in S} F_j(X))$ and $F \circ F(X)$.

- **Sierpinski gasket**

The Sierpinski gasket SG is a well-known examples of self-similar sets. Here are the first 3 steps of the construction of the Sierpinski gasket:



The space of the left-hand side X is an equilateral triangle in \mathbb{R}^2 . In the centre we have 3 equilateral triangles, the length of whose edges are a half of the ones of X . The similitudes F_1 , F_2 and F_3 are defined by the 3 triangles, and the similarity ratios of F_j are $\frac{1}{2}$. The right-hand side is the space $F \circ F(X)$. Then, we get an IFS $(X, S = \{1, 2, 3\}, \{F_j\}_{j \in S})$, and it gives rise to SG . Moreover, SG satisfies the open set condition. Namely, we can choose an open set $O = \text{int}(X)$, and we find that $\bigcup_{j \in S} F_j(O) \subset O$ and $F_i(O) \cap F_j(O) = \emptyset$ for any $i \neq j \in S$. Therefore, the Hausdorff dimension of SG is the root α given by the equation $\sum_{j \in S} (\frac{1}{2})^\alpha = 3 \cdot (\frac{1}{2})^\alpha = 1$, i.e., $\dim_H(SG) = \log_2 3$.

• Pinwheel fractal

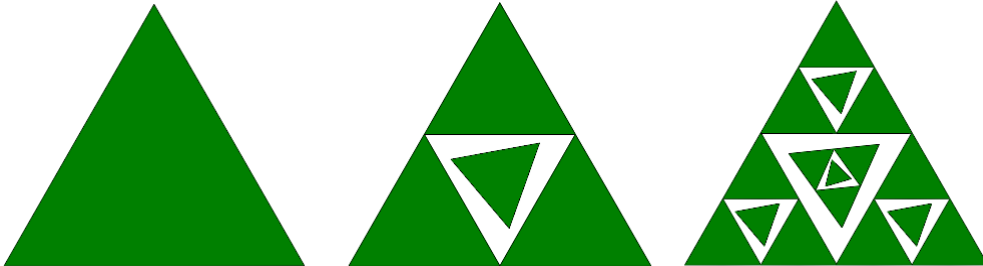
The Pinwheel fractal PW is a self-similar set which is modeled by the pinwheel tiling of the plane. There exist uncountably many pinwheel tilings, and therefore we have self-similar sets following them.



The figure is one of the pinwheel fractals based on the most well-known pinwheel tiling of \mathbb{R}^2 . The left triangle consists of 3 edges whose lengths are 1, 2 and $\sqrt{5}$, and we have 4 similitudes whose similarity ratios are $\frac{1}{\sqrt{5}}$. Therefore, we get an IFS $(X, S = \{1, \dots, 4\}, \{F_j\}_{j \in S})$ which gives rise to PW . Since PW satisfies the open set condition, the Hausdorff dimension of the pinwheel fractal is given by the root of the equation $\sum_{j \in S} (\frac{1}{\sqrt{5}})^\alpha = 4 \cdot (\frac{1}{\sqrt{5}})^\alpha = 1$, i.e., $\dim_H(PW) = \log_{\sqrt{5}} 4$.

• Infinite Sierpinski gasket

Here we give a non-connected self-similar set based on the Sierpinski gasket.



The row represents the first 3 iterations of an IFS that consists of 4 similitudes, one of which has the similarity ratio $\frac{1}{3}$ and the rest has $\frac{1}{2}$. The resulting cellular self-similar set consists of infinitely many countable connected components.

3 Main Theorem

In this section we define cyclic cocycles on certain subclass of self-similar sets and prove the main theorems. From now on, self-similar sets are assumed to be in \mathbb{R}^2 .

3.1 Cellular Self-similar Structures

First we define the kinds of self-similar sets on which we define cyclic cocycles.

Definition 3.1. Let $|X|$ be a 2-dimensional finite convex linear cell complex and $\{F_j\}_{j \in S}$ a set of similitudes $F_j : |X| \rightarrow |X|$ indexed by a finite set S . We also let $|X_1| = \bigcup_{j \in S} F_j(|X|)$. The triple $(|X|, S, \{F_j\}_{j \in S})$ is called a *cellular self-similar structure* if it satisfies

- a) $\partial|X| \subset \partial|X_1|$, and
- b) $\text{int} F_i(|X|) \cap \text{int} F_j(|X|) = \emptyset$, for all $i \neq j \in S$.

By Theorem 2.5 we have a unique self-similar set $K_{|X|}$ with respect to the cellular self-similar structure $(|X|, S, \{F_j\}_{j \in S})$ and we call $K_{|X|}$ the *cellular self-similar set with respect to* $(|X|, S, \{F_j\}_{j \in S})$. By construction, $K_{|X|}$ is a compact subset of $|X| \subset \mathbb{R}^2$.

Lemma 3.2. Any cellular self-similar structure $(|X|, S, \{F_j\}_{j \in S})$ satisfies the open set condition.

Proof. The lemma follows immediately from the definition of cellular self-similar structures. \square

Let $(|X|, S, \{F_j\}_{j \in S})$ be a cellular self-similar structure. For any $n \in \mathbb{N}$, we define a cell complex $|X_n|$ as follows: first, for $\omega = (j_1, \dots, j_n) \in S^{\times n}$, we write

$$F_\omega = F_{j_1} \circ \dots \circ F_{j_n}.$$

We define $|X_n|$ by the following skelton filtration:

- $sk_0(|X_n|) = \bigcup_{\omega \in S^{\times n}} F_\omega(sk_0(|X|))$,
- $sk_1(|X_n|) = \bigcup_{\omega \in S^{\times n}} F_\omega(sk_1(|X|))$,
- $sk_2(|X_n|) = \bigcup_{\omega \in S^{\times n}} F_\omega(sk_2(|X|)) = \bigcup_{\omega \in S^{\times n}} F_\omega(|X|)$.

A 1-cell in $|X_n|$ is defined to be the closure of a connected component in $sk_1(|X_n|) - sk_0(|X_n|)$. The definition of a cellular self-similar structure yields

$$|X_{n+1}| = \bigcup_{j \in S} F_j \left(\bigcup_{\omega \in S^{\times n}} F_\omega(|X|) \right) = \bigcup_{j \in S} F_j(|X_n|),$$

and therefore we have an inclusion map $i_{n,n+1} : |X_{n+1}| \hookrightarrow |X_n|$ for every $n \in \mathbb{Z}_{\geq 0}$. Moreover $K_{|X|}$ is written as the inverse limit of inclusion maps $\{i_{n,n+1} : |X_{n+1}| \hookrightarrow |X_n|\}$, that is,

$$K_{|X|} = \bigcap_{n=1}^{\infty} |X_n|.$$

Therefore we also have a canonical inclusion map $i_n : K_{|X|} \hookrightarrow |X_n|$ for each $n \in \mathbb{Z}_{\geq 0}$.

For a $n \in \mathbb{N}$ and a 1-cell $|\sigma|$ in $\partial|X_n|$, we define E_{σ}^n to be the set of 1-cells of $|X_{n+1}|$ which are subspaces of $|\sigma|$. Then, we have

$$|\sigma| = \bigcup_{|\tau| \in E_{\sigma}^n} |\tau|.$$

Lemma 3.3. *There exists $M \in \mathbb{N}$ that satisfy the following condition: for any $n \in \mathbb{N}$ and a 1-cell $|\sigma|$ in $\partial|X_n|$ we have $\#E_{\sigma}^n \leq M$.*

Proof. For every 1-cell $|\sigma|$ in $\partial|X_n|$, there exists a unique $\omega \in S^{\times n}$ and a unique 1-cell $|\tilde{\sigma}|$ in $F_{\omega}(|X|)$ such that $|\sigma| \subset |\tilde{\sigma}|$. Since $|X_{n+1}|$ is obtained by replacing each 2-cell $F_{\omega}(|X|)$ by $F_{\omega}(|X_1|) = F_{\omega}(\bigcup_{j \in S} F_j(|X|))$, $|\tilde{\sigma}|$ is subdivided by at most $\#S$ 2-cells. This completes the proof of the lemma. \square

Now, since $|X_n|$ is a convex linear cell complex, we can associate an abstract simplicial complex X_n^s by employing a lemma in [27]:

Lemma 3.4. [27, Chapter I, Lemma 1] *A convex linear cell complex can be subdivided into a simplicial complex without introducing any more vertices.*

For any simplicial complex $|X_n^s|$ and $p \geq 0$ we define $S_p(X_n^s)$ to be a set of $(p+1)$ -tuples of points of $sk_0(X_n^s)$ such that $(p+1)$ vertices are contained in a simplex of X_n^s , that is,

$$S_p(X_n^s) = \left\{ (x_0, \dots, x_p) \in sk_0(X_n^s)^{\times(p+1)} \mid \text{there exists a } p\text{-simplex } \sigma \in X_n^s \text{ s.t. } x_i \in \sigma \text{ for } \forall i \right\}.$$

We also define face maps $\sigma_i : S_p(X_n^s) \rightarrow S_{p-1}(X_n^s)$ for $0 \leq i \leq p$, and the pair $(S_*(X_n^s), \sigma_i)$ forms a semi-simplicial set; see the definition [6]. We note that, for $p \geq 1$, $S_p(X_n^s)$ contains a *degenerate simplex* (x_0, \dots, x_p) , that is, a simplex $(x_0, \dots, x_p) \in S_p(X_n^s)$ such that there exist distinct indexes i and j such that $x_i = x_j$. Now, we define $\tilde{S}_p(X_n^s; \mathbb{C})$ to be the free \mathbb{C} -module generated by $S_p(X_n^s)$ and a map $\tilde{\partial}_p : \tilde{S}_p(X_n^s; \mathbb{C}) \rightarrow \tilde{S}_{p-1}(X_n^s; \mathbb{C})$ by

$$\tilde{\partial}_p(x_0, \dots, x_p) = \sum_{j=0}^p (-1)^j \sigma_i(x_0, \dots, x_p) = \sum_{j=0}^p (-1)^j (x_0, \dots, \hat{x}_j, \dots, x_p).$$

Then we have a commutative diagram:

$$\begin{array}{ccc} \tilde{S}_p(X_n^s; \mathbb{C}) & \xrightarrow{\tilde{\partial}_p} & \tilde{S}_{p-1}(X_n^s; \mathbb{C}) \\ \pi \downarrow & & \downarrow \pi \\ C_p(X_n^s; \mathbb{C}) & \xrightarrow{\partial_p} & C_{p-1}(X_n^s; \mathbb{C}), \end{array}$$

where $C_p(X_n^s; \mathbb{C})$ is the p^{th} simplicial chain group of X_n^s whose coefficient is \mathbb{C} , ∂_p a simplicial boundary map and π the quotient map.

Remark 3.5. The chain map π is a chain equivalence; see Theorem 8 in Chapter 4.3 of [21] for details.

We now assign the counterclockwise orientation on each 2-simplex in every $|X_n^s|$, and choose a basis $B_n = \{[\sigma]\}$ of $C_2(X_n^s; \mathbb{C})$ consisting of non-degenerate p -simplexes σ in X_n^s . We assume that each element $[\sigma]$ of B_n represents the counterclockwise orientation.

Now, we define simplicial chains for every $n \in \mathbb{Z}_{\geq 0}$: let

$$c_n = \sum_{[\sigma] \in B_n} [\sigma] \in C_2(X_n^s; \mathbb{C}).$$

Then $\partial_2(c_n) \in C_1(X_n^s; \mathbb{C})$ is the sum of all 1-simplices which lie on $\partial|X_n^s|$, and we can choose $s_n \in \pi^{-1}(c_n)$ so that s_n has no degenerate simplexes and each summand of $\tilde{\partial}_2(s_n) \in \tilde{S}_1(X_n^s; \mathbb{C})$ lies on $\partial|X_n^s|$. Now we define a boundary chain $b_n \in \tilde{S}_1(X_n^s; \mathbb{C})$ by

- $b_n = \tilde{\partial}_2(s_n)$.

We first let $\mathcal{E}(b_n)$ be the subset of 1-simplices in $S_1(X_n^s)$ which are direct summands of b_n . Since any $\sigma \in \mathcal{E}(b_n)$ is non-degenerate, we can take the geometric realisation $|\sigma| \subset \partial|X_n^s|$. We also define a subset $\mathcal{E}(o_n) \subset \mathcal{E}(b_n)$ by

$$\mathcal{E}(o_n) = \left\{ \sigma \in \mathcal{E}(b_n) \mid |\sigma| \subset \partial|X| \right\}.$$

For each $\sigma \in \mathcal{E}(o_n)$, we have the sign of σ in b_n and denote it by $\text{sgn}(\sigma)$. Define

- $o_n = \sum_{\sigma \in \mathcal{E}(o_n)} \text{sgn}(\sigma) \cdot \sigma \in \tilde{S}_1(X_n^s; \mathbb{C})$,
- $I_n = b_n - o_n \in \tilde{S}_1(X_n^s; \mathbb{C})$.

Let $\mathcal{E}(I_n) = \mathcal{E}(b_n) \setminus \mathcal{E}(o_n)$. We also define $|\mathcal{E}(I_n)| = \bigcup_{\sigma \in \mathcal{E}(I_n)} |\sigma|$, and $\mathcal{E}(I_n \setminus I_{n-1})$ in a manner similar to $\mathcal{E}(o_n)$:

$$\mathcal{E}(I_n \setminus I_{n-1}) = \left\{ \sigma \in \mathcal{E}(b_n) \mid |\sigma| \subset \overline{|\mathcal{E}(I_n)| \setminus |\mathcal{E}(I_{n-1})|} \right\}.$$

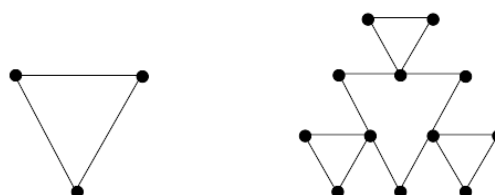
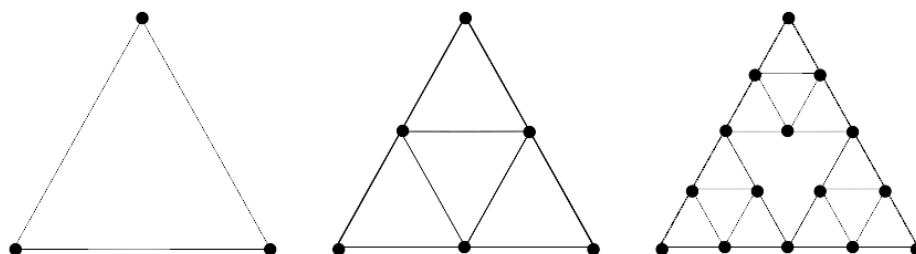
Finally we define a 1-chain by

- $I_n \setminus I_{n-1} = \sum_{\sigma \in \mathcal{E}(I_n \setminus I_{n-1})} \text{sgn}(\sigma) \cdot \sigma \in \tilde{S}_1(X_n^s; \mathbb{C})$.

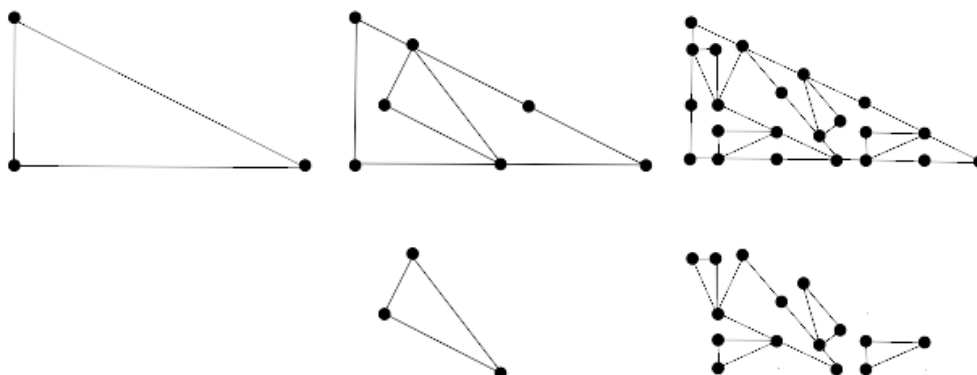
Example 3.6.

For the Sierpinski gasket and the pinwheel fractal introduced in Section 2.2, we give spaces that represent $\mathcal{E}(b_0)$, $\mathcal{E}(b_1)$, $\mathcal{E}(b_2)$, and $\mathcal{E}(I_0)$, $\mathcal{E}(I_1)$, $\mathcal{E}(I_2)$. The first row corresponds to $\mathcal{E}(b_i)$, and the second corresponds to $\mathcal{E}(I_i)$. The dots in spaces denote the vertices of 1-simplices, i.e., 0-simplices.

- Sierpinski gasket



- Pinwheel fractal



Next, for every $n \in \mathbb{Z}_{\geq 0}$, we define a 2-dimensional cell complex $|K_{n,n+1}|$. For every $n \in \mathbb{Z}_{\geq 0}$ we endow

$$|X_{n,n+1}| = \overline{|X_n| - |X_{n+1}|} \quad (= \text{the closure of } |X_n| - |X_{n+1}|),$$

with a cell complex structure, whose structure is defined by the following skelton filtration:

- $sk_0(|X_{n,n+1}|) = sk_0(\partial|X_{n+1}|) \cap |X_{n,n+1}|$
- $sk_1(|X_{n,n+1}|) = \partial|X_{n,n+1}|$
- $sk_2(|X_{n,n+1}|) = |X_{n,n+1}|$

We also define a subspace $|K_{n,n+1}|$ in \mathbb{R}^3 to be

$$|K_{n,n+1}| = [0, 1] \times \partial|X_{n+1}| \cup \{1\} \times |X_{n,n+1}|.$$

We use z as the variable of the first coordinate of $|K_{n,n+1}|$. We now endow $|K_{n,n+1}|$ with a 2-dimensional cell complex structure as follows: let $p_1 : |K_{n,n+1}| \rightarrow |K_{n,n+1}|_{|z=1}$ be a projection defined by $p_1(t, x) = (1, x)$. We define

- $sk_0(|K_{n,n+1}|) = \{0\} \times sk_0(\partial|X_{n+1}|) \cup \{1\} \times (sk_0(\partial|X_n|) \cup sk_0(|X_{n,n+1}|))$
- $sk_1(|K_{n,n+1}|) = \{0\} \times sk_1(\partial|X_{n+1}|) \cup \{1\} \times (sk_1(\partial|X_n|) \cup sk_1(\partial|X_{n,n+1}|) \cup |E_{n,n+1}|)$
- $sk_2(|K_{n,n+1}|) = |K_{n,n+1}|$.

Here,

$$|E_{n,n+1}| = \left\{ (x, y) \mid x \in \{1\} \times sk_0(\partial|X_n|) \text{ or } x \in \{1\} \times sk_0(|X_{n,n+1}|), \right. \\ \left. y \in \{0\} \times sk_1(\partial|X_{n+1}|) \text{ such that } p_1(x) = y \right\}.$$

By construction of $|K_{n,n+1}|$, we have

$$\partial|K_{n,n+1}| = \{0\} \times \partial|X_{n+1}| \cup \{1\} \times \partial|X_n|$$

as a cell complex in \mathbb{R}^3 . By employing Lemma 3.4 again, the cell complex $|K_{n,n+1}|$ is subdivided into a 2-dimensional simplicial complex $|K_{n,n+1}^s|$, and we may therefore choose chains $s_{n,n+1}$, $\tilde{s}_{n,n+1}$ and $\tilde{\tilde{s}}_{n,n+1} \in \tilde{S}_2(K_{n,n+1}^s; \mathbb{C})$ so that the chains consist of non-degenerate simplexes and

$$\tilde{\partial}_2(s_{n,n+1}) = b_n - b_{n+1}, \quad \tilde{\partial}_2(\tilde{s}_{n,n+1}) = I_n - I_{n+1}, \quad \tilde{\partial}_2(\tilde{\tilde{s}}_{n,n+1}) = I_{n+1} \setminus I_n.$$

We define the sets $\mathcal{E}(s_{n,n+1})$, $\mathcal{E}(\tilde{s}_{n,n+1})$ and $\mathcal{E}(\tilde{\tilde{s}}_{n,n+1})$ in a manner similar to the definition of $\mathcal{E}(b_n)$, and assume that $\tilde{s}_{n,n+1}$ and $\tilde{\tilde{s}}_{n,n+1}$ are summands of $s_{n,n+1}$, in other words,

$$\mathcal{E}(\tilde{s}_{n,n+1}), \mathcal{E}(\tilde{\tilde{s}}_{n,n+1}) \subset \mathcal{E}(s_{n,n+1}).$$

By a closed cycle z in $I_{n+1} \setminus I_n$ we mean a subset z of $\mathcal{E}(I_{n+1} \setminus I_n)$ such that $\bigcup_{\sigma \in z} |\sigma|$ is homomorphic to S^1 , and denote $\bigcup_{\sigma \in z} |\sigma|$ by $|z|$. We also denote by $\text{cyc}(I_{n+1} \setminus I_n)$ the set of closed

cycles in $I_{n+1} \setminus I_n$ and define $\tilde{z} = \sum_{\sigma \in z} \text{sgn}(\sigma) \cdot \sigma \in \tilde{S}_1(K_{n,n+1}^s; \mathbb{C})$ for $z \in \text{cyc}(I_{n+1} \setminus I_n)$. Then, for every closed cycle z in $I_{n+1} \setminus I_n$, there exists a non-degenerate 2-chain $\tilde{\tilde{z}} \in \tilde{S}_2(K_{n,n+1}^s; \mathbb{C})$ such that $\partial_2(\tilde{\tilde{z}}) = \tilde{z}$.

For $n = 0$ we define

$$|\tilde{K}_{0,1}| = [0, 1] \times \partial(|X| - |X_1|) \cup \{1\} \times |X_{0,1}|$$

and then $|\tilde{K}_{0,1}|$ is written as

$$|\tilde{K}_{0,1}| = \bigcup_{z \in \text{cyc}(I_1 \setminus I_0)} |\varepsilon(\tilde{\tilde{z}})|$$

since $\partial(|X| - |X_1|) = |\varepsilon(I_1 \setminus I_0)|$. Moreover, since, for every $\omega \in S^{\times n}$, we have an inclusion map $i_\omega : \partial(|X| - |X_1|) \hookrightarrow F_\omega(\partial|X_1|)$, there exists a family $\{\tilde{i}_\omega\}_{\omega \in S^{\times n}}$ of inclusion maps $\tilde{i}_\omega : |\tilde{K}_{0,1}| \hookrightarrow |K_{n,n+1}|$ such that

$$\tilde{i}_\omega|_{z=0} = i_\omega.$$

Finally we fix a subdivision of $|\tilde{K}_{0,1}|$ and assume that the subdivision of the images of the inclusion maps are given by the subdivision of $|\tilde{K}_{0,1}|$.

3.2 Cyclic Quasi-1-cocycle

In this subsection, we define a sequence of complex numbers for given Hölder continuous functions, that we call a cyclic quasi-1-cocycle. In order to define the sequence, we first recall a cochain complex which gives rise to one of the classical cohomology theories in algebraic topology, so called Alexander-Spanier cohomology theory; see Chapter 6 of [21] for details.

Let R be a ring. We also let X be a set and $X^{(p+1)}$ the $(p+1)$ -fold product of X . We define $F^p(X; R)$ to be the abelian group of functions from $X^{(p+1)}$ to R , whose sum is given by the pointwise sum. A coboundary homomorphism $\delta : F^p(X; R) \rightarrow F^{p+1}(X; R)$ is defined by

$$(\delta\phi)(x_0, \dots, x_{p+1}) = \sum_{j=0}^{p+1} (-1)^j \phi(x_0, \dots, \hat{x}_j, \dots, x_{p+1}).$$

We also introduce the cup product on the complex $(F^*(X; R), \delta)$: for $\phi_1 \in F^p(X; R)$ and $\phi_2 \in F^q(X; R)$ the cup product $\phi_1 \smile \phi_2 \in F^{p+q}(X; R)$ is defined by

$$(\phi_1 \smile \phi_2)(x_0, \dots, x_{p+q}) = \phi_1(x_0, \dots, x_p) \phi_2(x_p, \dots, x_{p+q}).$$

The Leibniz rule holds for the cup product: for $\phi_1 \in F^p(X; R)$ and $\phi_2 \in F^q(X; R)$,

$$\delta(\phi_1 \smile \phi_2) = \delta\phi_1 \smile \phi_2 + (-1)^p \phi_1 \smile \delta\phi_2.$$

Now, we define a cochain subcomplex of $(F^*(X; R), \delta)$: we assume that X is a metric space and R the field of complex numbers \mathbb{C} . We also let $C^\alpha(X)$ be the algebra of complex-valued α -Hölder continuous functions on X . Then, $C^\alpha(X)$ is a subalgebra of $F^0(X; \mathbb{C})$, and for each $p \in \mathbb{Z}_{\geq 0}$ we define the submodule $C^{\alpha,p}(X)$ of $F^p(X; \mathbb{C})$ generated by $C^\alpha(X) \subset F^0(X; \mathbb{C})$ with the coboundary maps and the cup product.

We now apply the construction for a cellular self-similar structure $(|X|, S, \{F_j\}_{j \in S})$: let $C^\alpha(K_{|X|})$ be the α -Hölder continuous functions defined on $K_{|X|}$. For each $n \in \mathbb{N}$, we endow

$sk_0(|X_n|)$ with the induced metric of \mathbb{R}^2 . Since we have an inclusion map $j_n : sk_0(|X_n|) \hookrightarrow K_{|X|}$ for every $n \in \mathbb{Z}_{\geq 0}$, we have a commutative diagram of cochain complexes

$$\begin{array}{ccc} F^p(K_{|X|}; \mathbb{C}) & \xrightarrow{j_n^*} & F^p(sk_0(|X_n^s|); \mathbb{C}) \\ \downarrow & & \downarrow \\ C^{\alpha,p}(K_{|X|}) & \xrightarrow{j_n^*} & C^{\alpha,p}(sk_0(|X_n^s|)) \end{array}$$

The vector space $F^p(sk_0(|X_n^s|); \mathbb{C})$ is the set of complex-valued functions $\text{Func}(S_p(\Delta^{\#sk_0(|X_n^s|)}), \mathbb{C})$ defined on $S_p(\Delta^{\#sk_0(|X_n^s|)}) := sk_0(|X_n^s|)^{\times p+1}$. In a manner similar to the definition of the face maps σ_i of $S_p(X_n^s)$, we define the face maps on $S_*(\Delta^{\#sk_0(|X_n^s|)})$, and then the pair $(S_*(\Delta^{\#sk_0(|X_n^s|)}), \sigma_i)$ turns out to be a semi-simplicial set, the definition of which is in [6], also known as the fundamental ∞ -groupoid of $sk_0(|X_n^s|)$. Since the inclusion map $S_*(|X_n^s|) \hookrightarrow S_*(\Delta^{\#sk_0(|X_n^s|)})$ is a map of semi-simplicial sets, we therefore get the following commutative diagram:

$$\begin{array}{ccccc} F^p(K_{|X|}; \mathbb{C}) & \xrightarrow{j_n^*} & F^p(sk_0(|X_n^s|); \mathbb{C}) & \xrightarrow[\text{linearly}]{\text{extend}} & \text{Hom}_{\mathbb{C}}(\tilde{S}_p(\Delta^{\#sk_0(|X_n^s|)}; \mathbb{C}), \mathbb{C}) \\ \downarrow & & \downarrow & & \downarrow \text{restrict} \\ C^{\alpha,p}(K_{|X|}) & \xrightarrow{j_n^*} & C^{\alpha,p}(sk_0(|X_n^s|)) & \xrightarrow{r} & \text{Hom}_{\mathbb{C}}(\tilde{S}_p(|X_n^s|; \mathbb{C}), \mathbb{C}). \end{array}$$

Now we define $C^{\alpha,p}(|X_n^s|) = \text{im}(r)$. For any $f, g \in C^{\alpha}(K_{|X|})$ and $p = 1$, we have a 1-cochain $\omega_n(f, g) = (f \smile \delta g) - (g \smile \delta f)$ in $C^{\alpha,1}(|X_n^s|)$ for every $n \in \mathbb{N}$. As we define in Section 3.1, we also have $I_n \in \tilde{S}_1(|X_n^s|; \mathbb{C})$. For every $n \in \mathbb{N}$, we have a complex number $\omega_n(f, g)(I_n)$ and denote it by $\phi_n(f, g)$.

Definition 3.7. Let $f, g \in C^{\alpha}(K_{|X|})$. We call the sequence $\{\phi_n(f, g)\}_{n \in \mathbb{N}}$ the *cyclic quasi-1-cocycle* for f and g .

3.3 Non-trivial Cyclic 1-cocycles

We prove the main results in this subsection. We refer the reader to [4, 5] for details of the Hochschild cohomology groups and the cyclic cohomology groups.

Theorem 3.8 (Existence theorem). *Let $(|X|, S, \{F_j\}_{j \in S})$ be a cellular self-similar structure with $\#S \geq 2$ and $K_{|X|}$ the cellular self-similar set with respect to $(|X|, S, \{F_j\}_{j \in S})$. We also let $C^{\alpha}(K_{|X|})$ be the algebra of complex-valued α -Hölder continuous functions on $K_{|X|}$. If $2\alpha > \dim_H(K_{|X|})$, then the cyclic quasi-1-cocycle $\{\phi_n(f, g)\}_{n \in \mathbb{N}}$ is a Cauchy sequence for any $f, g \in C^{\alpha}(K_{|X|})$.*

Proof. We first endow $|K_{n,n+1}|$ with a metric by $d((t, x), (t', x')) = |x - x'|_{\mathbb{R}^2}$. Let $f, g \in C^{\alpha}(K_{|X|})$. Since we have an inclusion map $sk_0(|X_{n+1}|) \hookrightarrow K_{|X|}$ for every $n \in \mathbb{Z}_{\geq 0}$, we can extend f to $f_n \in C^{\alpha}(sk_0(|K_{n,n+1}|))$ so that $f_n(t, x) = f(x)$ for $(t, x) \in [0, 1] \times \partial|X_{n+1}^s|$. We also let, for $h, k \in C^{\alpha}(sk_0(|K_{n,n+1}|))$, $\omega_n(h, k) = (h \smile \delta k) - (k \smile \delta h)$ be a 1-cochain in

$C^{\alpha,1}(|K_{n,n+1}^s|)$. Then, we have

$$\begin{aligned}
|\phi_n(f, g) - \phi_{n+1}(f, g)| &= |\omega_n(f_n, g_n)(I_n - I_{n+1})| \\
&= |\omega_n(f_n, g_n)(\tilde{\partial}_2(\tilde{s}_{n,n+1}))| \\
&\leq |\omega_n(f_n, g_n)(\tilde{\partial}_2(\tilde{s}_{n,n+1}))| + |\omega_n(f_n, g_n)(\tilde{\partial}_2(s_{n,n+1} - \tilde{s}_{n,n+1}))| \\
&\leq \sum_{\sigma \in \mathcal{E}(s_{n,n+1})} |\omega_n(f_n, g_n)(\tilde{\partial}_2(\sigma))| \\
&= \sum_{\sigma \in \mathcal{E}(s_{n,n+1})} |(\delta f_n \smile \delta g_n)(\sigma) - (\delta g_n \smile \delta f_n)(\sigma)|. \tag{1}
\end{aligned}$$

We note that every $\sigma \in \mathcal{E}(s_{n,n+1})$ is given by $\sigma = (x, y, z)$ for some $x, y, z \in sk_0(|K_{n,n+1}|)$. Therefore, (1) may be written as

$$\begin{aligned}
(1) &= \sum_{(x,y,z) \in \mathcal{E}(s_{n,n+1})} |(\delta f_n \smile \delta g_n)(x, y, z) - (\delta g_n \smile \delta f_n)(x, y, z)| \\
&= \sum_{(x,y,z) \in \mathcal{E}(s_{n,n+1})} \left| (f_n(y) - f_n(x))(g_n(z) - g_n(y)) - (g_n(y) - g_n(x))(f_n(z) - f_n(y)) \right| \\
&\leq \sum_{(x,y,z) \in \mathcal{E}(s_{n,n+1})} 2 \cdot c_f \cdot c_g |y - x|^\alpha |z - y|^\alpha, \tag{2}
\end{aligned}$$

where c_f and c_g are the Hölder constants of f and g respectively.

We now define a map to estimate the term (2). For any $\sigma \in \mathcal{E}(s_{n,n+1}) \setminus \mathcal{E}(\tilde{s}_{n,n+1})$ there exists a unique $\omega = (j_1, \dots, j_n) \in S^{\times n}$ such that $p_1(|\sigma|) \subset \partial F_\omega(|X|)$. We therefore have a map $\rho : \mathcal{E}(s_{n,n+1}) \setminus \mathcal{E}(\tilde{s}_{n,n+1}) \rightarrow S^{\times n}$, and define $\tilde{S}^{\times n}$ to be $\text{im}(\rho)$. We note that, by Lemma 3.3, there exists $M \in \mathbb{N}$ such that $\#\rho^{-1}(\omega) < M$ for any $\omega \in \tilde{S}^{\times n}$. Moreover, since $p_1(|\sigma|) \subset \partial F_\omega(|X|)$ we have an inequality

$$\text{diam}(|\sigma|) = \text{diam}(p_1(|\sigma|)) \leq r_{j_1} \cdots r_{j_n} \cdot d_{K_{|X|}},$$

where $(j_1, \dots, j_n) = \omega \in \tilde{S}^{\times n}$, r_j are the similarity ratios of F_j and $d_{K_{|X|}}$ is the diameter of $K_{|X|}$.

On the other hand, we let $L = \#\text{cyc}(I_1 \setminus I_0)$ be the number of closed cycles in $I_1 \setminus I_0$. At the $(n+1)$ -step, for every $\omega \in S^{\times n}$, there exist L closed cycles in $F_\omega(\bigcup_{j \in S} F_j(|X|)) = F_\omega(|X_1|)$. We recall that for every closed cycle z in $I_{n+1} \setminus I_n$ there is a 2-chain $\tilde{s}_z \in \tilde{S}_2(K_{n,n+1}^s)$ such that $\mathcal{E}(\tilde{s}_z) \subset \mathcal{E}(\tilde{s}_{n,n+1})$ and $\tilde{\partial}_2(\tilde{s}_z) = z$; see also Section 3.1. Therefore, $\tilde{s}_{n,n+1}$ may be written as

$$\tilde{s}_{n,n+1} = \sum_{\omega \in S^{\times n}} \sum_{1 \leq i \leq L} \tilde{s}_{\omega, z_i}.$$

We also recall from Section 3.1 that for every $\omega \in S^{\times n}$ we have an inclusion map $\tilde{i}_\omega : |\tilde{K}_{0,1}^s| \hookrightarrow |K_{n,n+1}|$ and

$$\text{im}(\tilde{i}_\omega) = \bigcup_{z \in \text{cyc}(I_{n+1} \setminus I_n) \text{ s.t. } |z| \subset F_\omega(|X_1|)} |\mathcal{E}(\tilde{s}_z)|.$$

Therefore, since the subdivision of the images $\text{im}(\tilde{i}_\omega)$ are induced by the subdivision of $|K_{0,1}^s|$,

we may define

$$\bar{M} = \sup_{z \in \text{cyc}(I_{n+1} \setminus I_n)} \{\#\mathcal{E}(\tilde{s}_z)\} = \sup_{z \in \text{cyc}(I_1 \setminus I_0)} \{\#\mathcal{E}(\tilde{s}_z)\}.$$

From these arguments, (2) is now decomposed into two parts:

$$\begin{aligned} (2) &= \sum_{(x,y,z) \in \mathcal{E}(s_{n,n+1}) \setminus \mathcal{E}(\tilde{s}_{n,n+1})} 2 \cdot c_f \cdot c_g |y-x|^\alpha |z-y|^\alpha \\ &\quad + \sum_{(x,y,z) \in \mathcal{E}(\tilde{s}_{n,n+1})} 2 \cdot c_f \cdot c_g |y-x|^\alpha |z-y|^\alpha \\ &\leq \sum_{(j_1, \dots, j_n) \in \tilde{S}^{\times n}} 2 \cdot c_f \cdot c_g \cdot \#\rho^{-1}(\omega) \cdot (r_{j_1}^{2\alpha} \dots r_{j_n}^{2\alpha} \cdot d_{K_{|X|}}^{2\alpha}) \\ &\quad + \sum_{(j_1, \dots, j_n) \in S^{\times n}} \sum_{1 \leq i \leq L} 2 \cdot c_f \cdot c_g \cdot \#\mathcal{E}(\tilde{s}_{z_i}) \cdot (r_{j_1}^{2\alpha} \dots r_{j_n}^{2\alpha} \cdot d_{K_{|X|}}^{2\alpha}) \\ &\leq \sum_{(j_1, \dots, j_n) \in S^{\times n}} 2 \cdot c_f \cdot c_g \cdot M \cdot (r_{j_1}^{2\alpha} \dots r_{j_n}^{2\alpha} \cdot d_{K_{|X|}}^{2\alpha}) \\ &\quad + \sum_{(j_1, \dots, j_n) \in S^{\times n}} 2 \cdot c_f \cdot c_g \cdot L \cdot \bar{M} \cdot (r_{j_1}^{2\alpha} \dots r_{j_n}^{2\alpha} \cdot d_{K_{|X|}}^{2\alpha}) \\ &= 2 \cdot c_f \cdot c_g \cdot d_{K_{|X|}}^{2\alpha} \cdot (M + L \cdot \bar{M}) \cdot \left(\sum_{j \in S} r_j^{2\alpha} \right)^n. \end{aligned}$$

We denote $2 \cdot c_f \cdot c_g \cdot d_{K_{|X|}}^{2\alpha} \cdot (M + L \cdot \bar{M})$ by K , and then we have

$$\begin{aligned} |\phi_{n+k}(f, g) - \phi_n(f, g)| &\leq \sum_{1 \leq i \leq k} |\phi_{n+i}(f, g) - \phi_{n+i-1}(f, g)| \\ &\leq \sum_{1 \leq i \leq k} K \cdot \left(\sum_{j \in S} r_j^{2\alpha} \right)^{n+i-1} \\ &= K \cdot \left(\sum_{j \in S} r_j^{2\alpha} \right)^n \cdot \sum_{1 \leq i \leq k} \left(\sum_{j \in S} r_j^{2\alpha} \right)^{i-1}. \end{aligned} \quad (3)$$

Since we assume that $2\alpha > \dim_H(K_{|X|})$ and $\dim_H(K_{|X|})$ is computed by the formula in Theorem 2.13, and therefore the term $(\sum_{j \in S} r_j^{2\alpha})$ is less than 1, and the term $\sum_{1 \leq i \leq k} (\sum_{j \in S} r_j^{2\alpha})^{i-1}$ converges to a finite value as k tends to ∞ . Therefore, we have

$$(3) \leq K \cdot \sum_{i=1}^{\infty} \left(\sum_{j \in S} r_j^{2\alpha} \right)^{i-1} \cdot \left(\sum_{j \in S} r_j^{2\alpha} \right)^n,$$

and the right hand side also converges to 0 as n tends to ∞ . This completes the proof of the theorem. \square

From now on, we assume that $2\alpha > \dim_H(K_{|X|})$ and define a bilinear map

$$\phi : C^\alpha(K_{|X|}) \times C^\alpha(K_{|X|}) \rightarrow \mathbb{C}$$

by $\phi(f, g) = \lim_{n \rightarrow \infty} \phi_n(f, g)$.

Lemma 3.9. *The map ϕ is independent of the choice of I_n .*

Proof. In order to check the mentioned property of the bilinear map $\phi : C^\alpha(K_{|X|}) \times C^\alpha(K_{|X|}) \rightarrow \mathbb{C}$, we have to show that the cyclic quasi-1-cocycle converges to the same value regardless of the choice of I_n which represents the given orientation. Let $I_n, I'_n \in \pi^{-1}([I_n])$ such that $|\varepsilon(I_n)| = |\varepsilon(I'_n)|$ and $\phi'_n(f, g) = (f \smile \delta g)(I'_n) - (g \smile \delta f)(I'_n)$. Then there exists a 2-dimensional simplicial complex J_n such that $|J_n| = |\varepsilon(I_n)| \times [0, 1]$, and we may choose $\hat{s}_n \in \tilde{S}_2(J_n; \mathbb{C})$ such that $\tilde{\partial}_2(\hat{s}_n) = I_n - I'_n$. We endow $|J_n|$ with a metric similar to the metric on $|K_{n,n+1}|$, and then we have

$$\begin{aligned}
|\phi_n(f, g) - \phi'_n(f, g)| &= |\delta \omega_n(f_n, g_n)(\hat{s}_n)| \\
&\leq 2 \sum_{(x,y,z) \in \varepsilon(\hat{s}_n)} c_f \cdot c_g \cdot |y-x|^\alpha \cdot |z-y|^\alpha \\
&\leq 2 \sum_{(x,y) \in \varepsilon(I_n)} 2 \cdot c_f \cdot c_g \cdot |y-x|^{2\alpha} \\
&\leq 2 \sum_{(j_1, \dots, j_n) \in \tilde{S}^{\times n}} 2 \cdot c_f \cdot c_g \cdot d_{K_{|X|}}^{2\alpha} \cdot r_{j_1}^{2\alpha} \cdot \dots \cdot r_{j_n}^{2\alpha} \\
&\leq 4 \cdot c_f \cdot c_g \cdot d_{K_{|X|}}^{2\alpha} \cdot (\sum_{j \in S} r_j^{2\alpha})^n \\
&\rightarrow 0, \text{ as } n \rightarrow \infty.
\end{aligned}$$

This completes the proof of the well-definedness of ϕ . \square

Based on the proof of Theorem 3.8, we can prove the following corollary.

Corollary 3.10. *For any $f, g \in C^\alpha(K_{|X|})$, we have*

$$\phi(f, g) = 2 \int_{\partial|X|} f dg = 2 \cdot (\text{Young integral along } \partial|X|).$$

In particular, for 1 and $x := id \in C^\alpha(K_{|X|})$,

$$\phi(1, x) = 2 \int_{\partial|X|} dx = 2 \cdot (\text{length of } \partial|X|).$$

Proof. By the construction of the cyclic quasi-1-cocycle of $f, g \in C^\alpha(K_{|X|})$, we have

$$\phi_n(f, g) = \omega_n(f, g)(I_n) = -\omega_n(f, g)(o_n) + \omega_n(f, g)(b_n).$$

The proof of Theorem 3.8 yields directly that the sequence $\{\omega_n(f, g)(b_n)\}_{n \in \mathbb{Z}_{\geq 0}}$ converges to 0 if $2\alpha > \dim_H(K_{|X|})$. Since $\{\omega_n(f, g)(o_n)\}_{n \in \mathbb{Z}_{\geq 0}}$ provides the Young integration along $\partial|X|$, which is the finite union of closed segments, we get the mentioned equalities. \square

Theorem 3.11. *Under the assumption of Theorem 3.8 :*

- a) *The bilinear map ϕ gives rise to a cyclic 1-cocycle of $C^\alpha(K_{|X|})$.*
- b) *If $|X| \neq |X_1|$, the cocycle ϕ represents a non-trivial element $[\phi]$ in $HC^1(C^\alpha(K_{|X|}))$.*

Proof. We have a linear map $\phi : C^\alpha(K_{|X|}) \otimes C^\alpha(K_{|X|}) \rightarrow \mathbb{C}$. It follows immediately that the cocycle satisfies the cyclic condition since $\phi_n(f, g)$ satisfies the cyclic condition for any $n \in \mathbb{Z}_{\geq 0}$. Accordingly, it remains to show that ϕ is a Hochschild cocycle. For $f, g, h \in C^\alpha(K_{|X|})$, we may write $b\phi(f, g, h)$ as

$$\begin{aligned} b\phi(f, g, h) &= \phi(fg, h) - \phi(f, gh) + \phi(hf, g) \\ &= \lim_{n \rightarrow \infty} \phi_n(fg, h) - \lim_{n \rightarrow \infty} \phi_n(f, gh) + \lim_{n \rightarrow \infty} \phi_n(hf, g) \\ &= \lim_{n \rightarrow \infty} \left(\phi_n(fg, h) - \phi_n(f, gh) + \phi_n(hf, g) \right) \\ &= \lim_{n \rightarrow \infty} b\phi_n(f, g, h). \end{aligned}$$

Therefore, to prove that $b\phi(f, g, h) = 0$ is equivalent to prove that $\lim_{n \rightarrow \infty} b\phi_n(f, g, h) = 0$. Using $\delta(\eta \smile \tau) = \delta\eta \smile \tau + (-1)^{\deg(\eta)} \eta \smile \delta\tau$, we have

$$\begin{aligned} \phi_n(fg, h) &= \left(fg \smile \delta h - h \smile \delta(fg) \right) (I_n) \\ &= \left((f \smile g \smile \delta h) - (h \smile \delta f \smile g) - (h \smile f \smile \delta g) \right) (I_n). \end{aligned}$$

Similarly,

$$\begin{aligned} \phi_n(f, gh) &= \left((f \smile \delta g \smile h) + (f \smile g \smile \delta h) - (g \smile h \smile \delta f) \right) (I_n), \\ \phi_n(hf, g) &= \left((h \smile f \smile \delta g) - (g \smile \delta h \smile f) - (g \smile h \smile \delta f) \right) (I_n). \end{aligned}$$

Therefore,

$$b\phi_n(f, g, h) = - \left((h \smile \delta f \smile g) + (f \smile \delta g \smile h) + (g \smile \delta h \smile f) \right) (I_n). \quad (4)$$

Since

$$\begin{aligned} -(h \smile \delta f \smile g)(I_n) &= \left((\delta h \smile f \smile g) + (h \smile f \smile \delta g) - (\delta(hfg)) \right) (I_n) \\ &= \left((\delta h \smile f \smile g) + (h \smile f \smile \delta g) \right) (I_n), \end{aligned}$$

we have

$$\begin{aligned} (4) &= \left((\delta h \smile f \smile g) + (h \smile f \smile \delta g) - (f \smile \delta g \smile h) - (g \smile \delta h \smile f) \right) (I_n) \\ &= \sum_{(x, y) \in \mathcal{E}(I_n)} \pm \left((\delta h \smile f \smile g) + (h \smile f \smile \delta g) - (f \smile \delta g \smile h) - (g \smile \delta h \smile f) \right) (x, y) \\ &= \sum_{(x, y) \in \mathcal{E}(I_n)} \pm (h(y) - h(x))(g(y) - g(x))(f(y) - f(x)). \end{aligned}$$

Therefore

$$\begin{aligned}
|b\phi_n(f, g, h)| &\leq \sum_{(x, y) \in \mathcal{E}(I_n)} |h(y) - h(x)| \cdot |g(y) - g(x)| \cdot |f(y) - f(x)| \\
&= \sum_{(x, y) \in \mathcal{E}(I_n)} c_f \cdot c_g \cdot c_h \cdot |x - y|^{3\alpha} \\
&\leq c_f \cdot c_g \cdot c_h \cdot d_{K|X|}^{3\alpha} \cdot M \sum_{(j_1, \dots, j_n) \in S^{\times n}} r_{j_1}^{3\alpha} \cdot \dots \cdot r_{j_n}^{3\alpha} \\
&= c_f \cdot c_g \cdot c_h \cdot d_{K|X|}^{3\alpha} \cdot M \cdot \left(\sum_{j \in S} r_j^{3\alpha} \right)^n \\
&\rightarrow 0, \text{ as } n \rightarrow \infty.
\end{aligned}$$

This completes the proof of (a).

We now prove (b). We note that we have the pairing

$$HH_1(C^\alpha(K_{|X|})) \times HH^1(C^\alpha(K_{|X|})) \rightarrow \mathbb{C}.$$

As seen in Theorem 3.10, we know that $\phi(1 \otimes x) \neq 0$, and this completes the proof of (b). \square

Remark 3.12. The algebra of α -Hölder continuous functions on a compact metric space admits a Banach topology and it turns out to be a Banach algebra. However, we do not know whether or not the cocycle of Theorem 3.11 is continuous in the sense of a map between Banach algebras.

3.4 Examples

We examine the cyclic cocycle on some cellular self-similar sets. The spaces on which the cocycles are examined are the examples given in Section 2.2.

- **Sierpinski gasket**

The theorems in the Section 3.3 may be applied to the Sierpinski gasket SG , and the cyclic 1-cocycle ϕ on $C^\alpha(SG)$ is well-defined for $2\alpha > \dim_H(SG) = \log_2 3$. Moreover, the cocycle is non-trivial since $|X| \neq |X_1|$.

- **pinwheel fractal**

Pinwheel fractal PF may also be seen as a cellular self-similar set. The self-similar structure consists of 4 similitudes whose ratios are $\frac{1}{\sqrt{5}}$, see also section 2. The cyclic cocycle is well-defined if $2\alpha > \dim_H(PF) = \log_{\sqrt{5}} 4$ and non-trivial in HC^1 .

- **Infinite isolated Sierpinski gaskets**

The second example in Section 2.2 is a cellular self-similar structure, and we denote by ISG the resulting cellular self-similar set. The cyclic 1-cocycle may be defined on the space, and the cocycle is non-trivial. From now, we also discuss the structure of $HC^0(C^{Lip}(ISG))$.

By the self-similar structure of ISG , $\pi_0(ISG) = \bigoplus_{p \in \mathbb{N}} \mathbb{Z}$, each of whose summands corresponds to a connected component Y_p of ISG . Therefore ISG may be written as

$$ISG = \bigsqcup_{p \in \mathbb{N}} Y_p.$$

Then we have the canonical inclusion map

$$in_p : Y_p \rightarrow \bigsqcup_{p \in \mathbb{N}} Y_p = ISG$$

for any $p \in \mathbb{N}$. We now fix a base point $y_p \in Y_p$ for each $p \in \mathbb{N}$, and define a cyclic 0-cocycle ψ_p of $C^{Lip}(Y_p)$ by taking the value of y_p for any $f \in C^{Lip}(Y_p)$. Therefore, the canonical inclusion map in_p induces the map of cyclic cohomology groups:

$$(in_p)^* : HC^0(C^{Lip}(Y_p)) \rightarrow HC^0(C^{Lip}(ISG)).$$

We now let P be a finite subset of \mathbb{N} and assume that $\Psi_P = \sum_{p \in P} \alpha_p(in_p)_*([\psi_p]) = 0$. We also define $c_p \in C^{Lip}(ISG)$ by

$$c_p(y) = \begin{cases} 1, & y \in Y_p \\ 0, & \text{otherwise.} \end{cases}$$

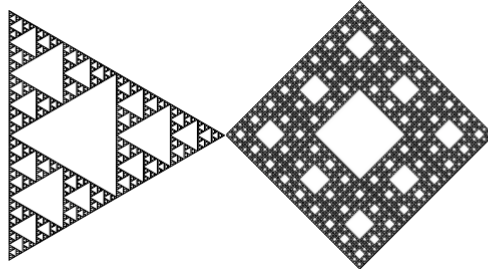
Then, for any $\tilde{p} \in P$, we have a pairing of the Hochschild homology group and the Hochschild cohomology group of $C^{Lip}(ISG)$:

$$0 = \langle \Psi_P, c_p \rangle = \sum_{p \in P} \alpha_p(in_p)_*([\psi_p])(c_{\tilde{p}}) = \alpha_{\tilde{p}},$$

and which means that the set $\{(in_p)_*([\psi_p])\}_{p \in P}$ is a linearly independent set. Since this argument also works for any finite set P of \mathbb{N} , we can conclude that $\{(in_p)_*([\psi_p])\}_{p \in \mathbb{N}}$ forms a linearly independent set of $HC^0(C^{Lip}(ISG))$, and therefore $HC^0(C^{Lip}(ISG))$ contains $\bigoplus_{p \in \mathbb{N}} \mathbb{C}$ as a \mathbb{C} -vector space.

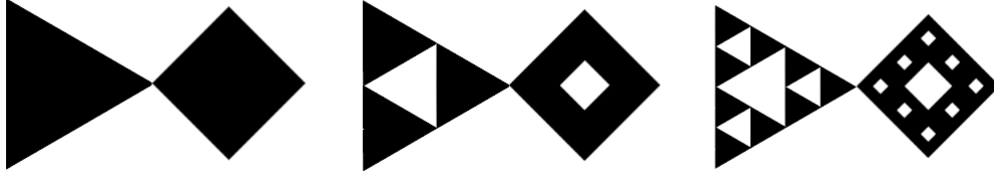
3.5 Further work

Strichartz introduces the notion of “fractafold” [22, 23], and on which he examines fractal versions of the classical theories, for example, Hodge-de Rham theory, spectral theory, homotopy theory. In particular, the Laplacian on some kinds of self-similar sets has been extensively studied, and it is applied to various fields [2, 12, 22, 23]. Here, we will give some examples of finite unions of cellular self-similar sets.

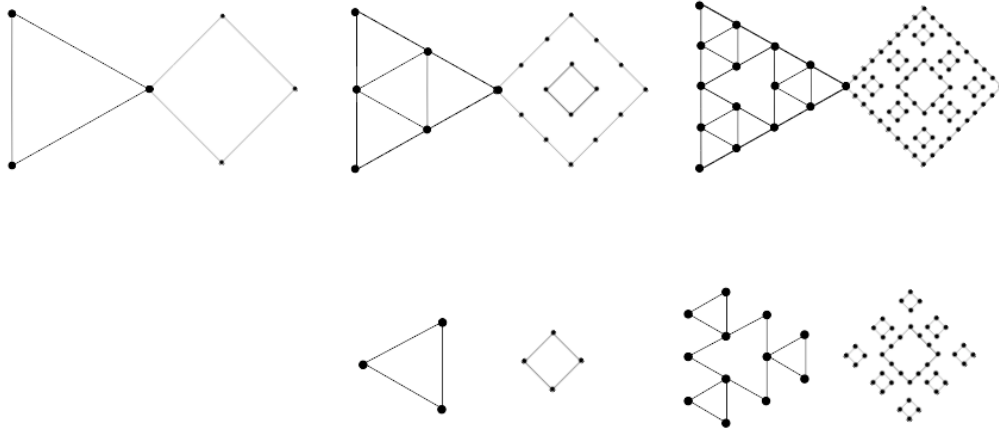


The first example is the wedge sum of Sierpinski gasket and Sierpinski carpet with base points at their corners. Then the space is neither a cellular self-similar set nor a fractafold. However,

the theorem may be applied to the space. Namely, the space is seen as the projective limit of the following spaces:



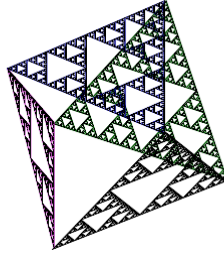
The figure is obtained by taking the wedge sum of the sequences which give rise to Sierpinski gasket and Sierpinski carpet. Similarly, we have sequences of boundary chains b_0, b_1, b_2 and inner chains I_0, I_1, I_2 respectively:



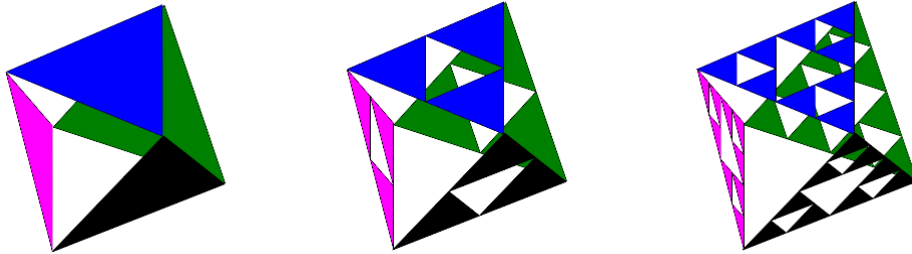
We therefore have a cyclic quasi-1-cocycle, and the quasi-cocycle can be written by the element-wise sum of cyclic quasi-1-cocycles of SG and SC . In order that that cyclic quasi-1-cocycle is a Cauchy sequence, it is enough that the Hölder index α satisfies the inequality $2\alpha > \dim_H(SC)$.

From the point of this view, SG can be seen as a union of 3 Sierpinski gaskets, and therefore SG may be seen as a “fractafold with boundary”, see [22, 23] for details. As defined in the previous subsection, we have a cyclic cocycle on SG .

Finally, we will define a cyclic cocycle of the algebra of Lipschitz functions defined on a “fractafold” based on the Sierpinski gasket:



The space is a union of four copies of Sierpinski gasket in \mathbb{R}^3 obtained by gluing the points at corners of a copy with each corner of the other Sierpinski gaskets. This space is one of the examples of what Strichartz calls “fractafolds without boundaries”, and we denote it by FSG . The space FSG can be seen as the projective limit of a sequence of the spaces that is obtained by gluing copies of the sequence which gives rise to SG .



We therefore get, by applying the theorem to each Sierpinski gasket, a cyclic 1-cocycle on $C^\alpha(FSG)$ when $2\alpha > \log_2 3$.

Remark 3.13. Strichartz introduces the Hodge-de Rham theory for fractal graphs [1]. In this paper, Laplacian on some fractal sets are defined by exploiting the Alexander-Spanier cochain complexes. However, we do not know whether or not there exist any relation between the cyclic 1-cocycle defined in the present paper and the Laplacian of [1].

References

- [1] S. Aaron, Z. Conn, R. Strichartz, H. Yu, *Hodge - de Rham theory on fractal graphs and fractals*, Commun. Pure Appl. Anal. 13 (2014), no. 2, p. 903-928.
- [2] M. L. Barlow, *Random walks and diffusions on fractals*, Proceedings of the International Congress of Mathematicians, Vol. I, II (Kyoto, 1990), 1025-1035, Math. Soc. Japan, Tokyo, 1991.
- [3] E. Christensen, C. Ivan, E. Schrohe, *Spectral triples and the geometry of fractals*, J. Noncommut. Geom. 6 (2012), no. 2, p.249-274.
- [4] A. Connes, *Non-commutative differential geometry*, Publications Mathématiques de l’I.H.É.S., tome 62, n° 2, 1985, p. 41-144.
- [5] A. Connes, *Noncommutative Geometry*, Academic Press, Inc., San Diego, CA, 1994.

- [6] S. Eilenberg, J. A. Zilber, *Semi-simplicial complexes and singular homology*, Annals of Mathematics, Second Series, Vol. 51, No. 3, 1950, p. 499-513.
- [7] K. Falconer, *Fractal geometry*, Mathematical foundations and applications, Second edition, John Wiley and Sons, Inc., Hoboken, NJ, 2003.
- [8] A. Grothendieck, *Produits tensoriels topologiques et espaces nucléaires*, Memoirs of the American Mathematical Society, No. 16, 1966.
- [9] N. Higson, J. Roe, *Analytic K-homology*, Oxford university press, 2000.
- [10] G. Hochschild, B. Kostant, A. Rosenberg, *Differential forms on regular affine algebras*, Transactions AMS 102, No.3, 1962, p. 383-408.
- [11] J. E. Hutchinson, *Fractals and self similarity*, Indiana Unit. Math. J. 30, 1981, p. 713-747.
- [12] J. Kigami, *Analysis on Fractals*, Cambridge Tracts in Mathematics (No. 143), Cambridge University Press, 2001.
- [13] J. L. Loday, *Cyclic homology*, Grundlehren der mathematischen Wissenschaften, Volume 301, 1998.
- [14] J. L. Loday, D. Quillen, *Cyclic homology and the Lie algebra homology of matrices*, Comment. Math. Helvetic 59, 1984, p. 565-591.
- [15] R. MacPherson, B. Schweinhurt, *Measuring shape with topology*, Journal of Mathematical Physics, 2012.
- [16] B. B. Mandelbrot, *The fractal geometry of nature*, W. H. Freeman and Co., San Francisco, Calif., 1982.
- [17] P. A. P. Moran, *Additive functions of intervals and Hausdorff measure*, Mathematical Proceedings of the Cambridge Philosophical Society, Volume 42, Issue 01, 1946, p. 15-23.
- [18] H. Moriyoshi, T. Natsume, *in preparation*.
- [19] J. Rosenberg, *Comparison between algebraic and topological K-theories for Banach algebras and C^* -algebras*, Handbook of K-theory. Vol. 1, 2, p. 843-874, Springer, Berlin, 2005.
- [20] J. Rosenberg, *The algebraic K-theory of operator algebras*, K-theory, 12(1), 1997, p. 75-99.
- [21] E. H. Spanier, *Algebraic Topology*, Springer New York, 1966.
- [22] R. Strichartz, *Fractafolds based on the Sierpinski gasket and their spectra*, Transactions of the American mathematical society, vol. 355, Num. 10, pp 4019-4043.
- [23] R. Strichartz, *Spectral analysis on infinite Sierpinski fractafolds*, arXiv:1011.1049v1 [math.FA] 4 Nov 2010.

- [24] B. L. Tsygan, *Homology of matrix Lie algebras over rings and the Hochschild homology*, Translation in Russ. Math. Survey 38(2) (1983), p.198-199.
- [25] S. Winter, *Minkowski content and fractal curvatures of self-similar tilings and generator formulas for self-similar sets*, Advances in Mathematics 274, 2015, p. 285-322.
- [26] L. C. Young, *An inequality of the Hölder type, connected with Stieltjes integration*, Acta Mathematica, Volume 67, issue 1, 1936, p. 251-282.
- [27] E. C. Zeeman, *Seminar on combinatorial topology*, Institut de Hautes Etudes Scientifiques, 1963.

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